Math 2040 C Week 12

$\boxed{\text{Let } V \text{ be an inner product space, } \mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}}$

<u>Recall</u> For a linear operator T on a finite dim. vector space,

T is diagonalizable $\Longleftrightarrow$ T has an eigenbasis

<u>Q</u> If $T \in L(V)$, when does T have an orthonormal eigenbasis?

<u>eg</u> $T: \mathbb{F}^2 \to \mathbb{F}^2$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \theta \neq k\pi, \ k \in \mathbb{Z}$$

<u>Ex</u> T has no eigenvalue for $\mathbb{F} = \mathbb{R}$

For $\mathbb{F} = \mathbb{C}$,

e.values	e.vectors
$\lambda_1 = e^{i\theta}$	$e_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} i \\ 1 \end{bmatrix}$
$\lambda_2 = e^{-i\theta}$	$e_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -i \\ 1 \end{bmatrix}$

orthonormal

$\therefore$ T has orthonormal eigenbasis if $\mathbb{F} = \mathbb{C}$
no orthonormal eigenbasis if $\mathbb{F} = \mathbb{R}$

<u>eg</u>

$$S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

eigenvalues $1, 2$

$$E(1, S) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$$

$$E(2, S) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$$

not orthogonal

$\therefore$ S is diagonalizable but does not have orthonormal eigenbasis

Self Adjoint and Normal Operator

Defn 7.11, 7.18 Let $T \in L(V)$.

T is called self-adjoint if $T = T^*$.

T is called normal if $TT^* = T^*T$

Rmk ① self-adjoint $\Rightarrow$ normal

② T is self-adjoint $\Longleftrightarrow$ $\langle T(v), w\rangle = \langle v, T(w)\rangle$ $\forall v, w \in V$

③ Self-adjoint operators in $L(V)$ $\xleftrightarrow{\text{analog}}$ Real numbers in $\mathbb{C}$

Spectral Theorem (proved later)

$T \in L(V)$ has orthonormal eigenbasis

$$\Longleftrightarrow \begin{cases} T \text{ is normal} & (\text{if } \mathbb{F} = \mathbb{C}) \\ T \text{ is self-adjoint} & (\text{if } \mathbb{F} = \mathbb{R}) \end{cases}$$

Prop 7.14 Let $\mathbb{F} = \mathbb{C}$, $T \in L(V)$.

If $\langle T(v), v\rangle = 0$ $\forall v \in V$,

then $T = T_0$ (the zero operator)

Pf Let $u, w \in V$. Then

$$\begin{aligned} 0 &= \langle T(u+w), u+w\rangle \\ &= \langle T(u) + T(w), u+w\rangle \\ &= \langle T(u), u\rangle + \langle T(w), u\rangle + \langle T(u), w\rangle \\ &\quad + \langle T(w), w\rangle \end{aligned}$$

$\therefore$ $\langle T(w), u\rangle + \langle T(u), w\rangle = 0$ $\cdots$ ①

Similarly, $\langle T(u+iw), u+iw\rangle = 0$

$\Rightarrow$ $\langle T(iw), u\rangle + \langle T(u), iw\rangle = 0$

$\Rightarrow$ $i\langle T(w), u\rangle - i\langle T(u), w\rangle = 0$ $\cdots$ ②

① and ② $\Rightarrow \langle T(u), w\rangle = 0$

w is arbitrary : Put $w = T(u)$

$\Rightarrow \langle T(u), T(u)\rangle = 0$

$\Rightarrow T(u) = \vec{0} \quad \forall u \in V$

$\Rightarrow T = T_0$

<u>Rmk</u> If $\mathbb{F} = \mathbb{R}$,

$\langle T(v), v\rangle = 0 \ \ \forall v \in V \not\Longrightarrow T = T_0$

Counter-example: $T : \mathbb{R}^2 \to \mathbb{R}^2$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -y \\ x \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

T is anti-clockwise rotation by $\frac{\pi}{2}$

For $\mathbb{F} = \mathbb{R}$, an extra assumption on T is needed for similar result

$\boxed{\underline{\text{Prop 7.16}} \text{ Let } \mathbb{F} = \mathbb{R},\ T \in L(V),\ T = T^*. \text{ If } \langle T(v), v\rangle = 0\ \forall v \in V, \text{ then } T = T_0}$

<u>Pf</u> Pf is similar to that of 7.14.

For any $u, w \in V$,

$$\begin{aligned} 0 &= \langle T(u+w), u+w\rangle \\ &= \langle T(u) + T(w), u + w\rangle \\ &= \langle T(u), u\rangle + \langle T(w), u\rangle + \langle T(u), w\rangle \\ &\quad + \langle T(w), w\rangle \\ &= \langle w, T(u)\rangle + \langle T(u), w\rangle \qquad (\because T = T^*) \\ &= 2\langle T(u), w\rangle \end{aligned}$$

$\Rightarrow \langle T(u), w\rangle = 0$ for any $u, w \in V$

$\Rightarrow T = T_0$

Prop 7.20 Let $T \in L(V)$. Then

T is normal $\Leftrightarrow \|T(v)\| = \|T^*(v)\| \ \forall v \in V$

Pf Note that

① $\|T(v)\|^2 = \langle T(v), T(v)\rangle = \langle T^*T(v), v\rangle$

$\|T^*(v)\|^2 = \langle T^*(v), T^*(v)\rangle = \langle TT^*(v), v\rangle$

② $$\begin{aligned}(T^*T - TT^*)^* &= (T^*T)^* - (TT^*)^* \\ &= T^*(T^*)^* - (T^*)^* T^* \\ &= T^*T - TT^*\end{aligned}$$

$\therefore T^*T - TT^*$ is self-adjoint

$\therefore T$ is normal

$\Leftrightarrow T^*T - TT^* = T_0$

$\Leftrightarrow \langle (T^*T - TT^*)(v), v\rangle = 0 \ \forall v \in V \ \begin{pmatrix} 7.14 \\ 7.16 \end{pmatrix}$

$\Leftrightarrow \langle T^*T(v), v\rangle = \langle TT^*(v), v\rangle \ \forall v \in V$

$\Leftrightarrow \|T(v)\| = \|T^*(v)\|$

Prop 7.21 Suppose $T \in L(V)$ is normal.

If v is an e.vector of T with e.value λ

then v is an e.vector of T^* with e.value $\bar{\lambda}$

Pf First, note that $T - \lambda I$ is normal:

$$\begin{aligned}(T-\lambda I)^*(T-\lambda I) &= (T^* - \bar{\lambda} I)(T - \lambda I) \\ &= T^*T - \bar{\lambda} T - \lambda T^* + \lambda\bar{\lambda} I \\ &= TT^* - \bar{\lambda} T - \lambda T^* + \lambda\bar{\lambda} I \\ &= (T - \lambda I)(T^* - \bar{\lambda} I) \\ &= (T - \lambda I)(T - \lambda I)^*\end{aligned}$$

$$\begin{aligned}\therefore \|(T^* - \bar{\lambda} I)(v)\| &= \|(T - \lambda I)^*(v)\| \\ &= \|(T - \lambda I)(v)\| \quad (7.20) \\ &= \|\vec{0}\| = 0\end{aligned}$$

$\Rightarrow T^*(v) = \bar{\lambda} v, \ v \neq 0$

Prop 7.22 Suppose $T \in L(V)$ is normal, v_i is e.vector of T with e.value λ_i, $i=1,2$, $\lambda_1 \neq \lambda_2$. Then $v_1 \perp v_2$

Pf By 7.21, $T^*(v_2) = \overline{\lambda_2} v_2$

$$\begin{aligned} \therefore \ \lambda_1 \langle v_1, v_2 \rangle &= \langle \lambda_1 v_1, v_2 \rangle \\ &= \langle T(v_1), v_2 \rangle \\ &= \langle v_1, T^*(v_2) \rangle \\ &= \langle v_1, \overline{\lambda_2} v_2 \rangle \\ &= \lambda_2 \langle v_1, v_2 \rangle \end{aligned}$$

$\lambda_1 \neq \lambda_2 \Rightarrow \langle v_1, v_2 \rangle = 0$

Thm 7.24 (Complex Spectral Theorem) Let $\mathbb{F} = \mathbb{C}$, $\dim V < \infty$, $T \in L(V)$.

Then T is normal

$\Leftrightarrow$ T has an orthonormal eigenbasis

Pf ($\Leftarrow$)

Let $\beta = \{e_1, \cdots, e_n\}$ be orthonormal e.basis.

with $T(e_i) = \lambda_i e_i$

Let $A = M(T, \beta) = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

Then $M(T^*, \beta) = A^* = \begin{bmatrix} \overline{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \overline{\lambda_n} \end{bmatrix}$

$AA^* = AA^* = \begin{bmatrix} |\lambda_1|^2 & & 0 \\ & \ddots & \\ 0 & & |\lambda_n|^2 \end{bmatrix} \Rightarrow TT^* = T^*T$

($\Rightarrow$) Suppose T is normal

By Schur's Thm (6.38), $\exists$ orthonormal basis

$\beta = \{e_1, \cdots, e_n\}$ s.t. $M(T, \beta)$ is upper triangular.

Let $M(T, \beta) = A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ \bigcirc & & & a_{nn} \end{bmatrix}$

Then $M(T^*, \beta) = A^* = \begin{bmatrix} \overline{a_{11}} & & & \bigcirc \\ \overline{a_{12}} & \overline{a_{22}} & & \\ \vdots & \vdots & \ddots & \\ \overline{a_{1n}} & \overline{a_{2n}} & \cdots & \overline{a_{nn}} \end{bmatrix}$

Suffice to show A is diagonal

Note $T(e_1) = a_{11} e_1$

$T^*(e_1) = \overline{a_{11}} e_1 + \overline{a_{12}} e_2 + \cdots + \overline{a_{1n}} e_n$

T is normal, e_1 is e.vector of T

$\Rightarrow T^*(e_1) = \overline{a_{11}} e_1$ (By 7.21)

$\Rightarrow a_{12} = a_{13} = \cdots = a_{1n} = 0$

$\therefore A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ & a_{22} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ \bigcirc & & & a_{nn} \end{bmatrix}$

$T(e_2) = a_{22} e_2$

$T^*(e_2) = \overline{a_{22}} e_2 + \cdots + \overline{a_{2n}} e_n$

$\therefore e_2$ is e.vector of T

$\Rightarrow T^*(e_2) = \overline{a_{22}} e_2$ (By 7.21)

$\Rightarrow a_{23} = a_{24} = \cdots = a_{2n} = 0$

Inductively we can prove

$a_{ij} = 0$ for any $i < j$

$\therefore A$ is diagonal

β is an orthonormal eigenbasis of T

Next, we focus on self-adjoint operators

Prop 7.13

Eigenvalues of self-adjoint operators are real.

Pf let $T \in L(V)$, $T^* = T$

Let v be e.vector of e.value λ

Then
$$\begin{aligned} \lambda \langle v, v \rangle &= \langle \lambda v, v \rangle \\ &= \langle T(v), v \rangle \\ &= \langle v, T(v) \rangle \\ &= \langle v, \lambda v \rangle \\ &= \overline{\lambda} \langle v, v \rangle \end{aligned}$$

$v \neq \vec{0} \Rightarrow \langle v, v \rangle \neq 0$,

so, $\lambda = \overline{\lambda} \Rightarrow \lambda$ is real.

Prop 7.15 let $\mathbb{F} = \mathbb{C}$, $T \in L(V)$. Then

T is self-adjoint $\Leftrightarrow \langle T(v), v \rangle \in \mathbb{R} \ \forall v \in V$

Pf For any $T \in L(V)$, $v \in V$,

$$\begin{aligned} &\langle T(v), v \rangle - \overline{\langle T(v), v \rangle} \\ &= \langle T(v), v \rangle - \langle v, T(v) \rangle \\ &= \langle T(v), v \rangle - \langle T^*(v), v \rangle \\ &= \langle (T - T^*)(v), v \rangle \end{aligned}$$

$\therefore \quad T = T^*$

$\Leftrightarrow \langle (T - T^*)(v), v \rangle = 0 \ \forall v \in V \quad (7.14)$

$\Leftrightarrow \langle T(v), v \rangle - \overline{\langle T(v), v \rangle} = 0 \ \forall v \in V$

$\Leftrightarrow \langle T(v), v \rangle \in \mathbb{R} \ \forall v \in V$

Prop 7.28 Suppose $T \in L(V)$ is self-adjoint

$U \subseteq V$ is a T-invariant subspace. Then

1. $U^\perp$ is T-invariant
2. $T|_U \in L(U)$ is self-adjoint
3. $T|_{U^\perp} \in L(U^\perp)$ is self-adjoint

Pf 1. Suppose $v \in U^\perp$. For any $u \in U$,

$$\langle T(v), u \rangle = \langle v, T^*(u) \rangle = \langle v, T(u) \rangle$$

U is T-invariant $\Rightarrow T(u) \in U$

$v \in U^\perp \Rightarrow \langle T(v), u \rangle = \langle v, T(u) \rangle = 0$

$\therefore T(v) \in U^\perp \Rightarrow U^\perp$ is T-invariant

For 2, if $u, v \in U$, then

$$\begin{aligned} \langle T|_U(u), v \rangle &= \langle T(u), v \rangle \\ &= \langle u, T(v) \rangle \\ &= \langle u, T|_U(v) \rangle \end{aligned}$$

$\Rightarrow T|_U$ is self-adjoint

3 follows from 1 and 2

Some Fact about polynomials

① For $q(t) = t^2 + bt + c \in P_2(\mathbb{R})$

$q(t)$ is irreducible $\iff b^2 - 4c < 0$

② If $p(t) \in P(\mathbb{R})$, $\alpha \in \mathbb{C}$, then

$$p(\alpha) = 0 \iff p(\bar{\alpha}) = 0$$

Pf Let $p(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_0$

where each $a_i \in \mathbb{R}$

$$\begin{aligned} p(\alpha) = 0 &\iff a_n \alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_0 = 0 \\ &\iff \overline{a_n \alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_0} = \bar{0} = 0 \\ &\iff \overline{a_n}\,\overline{\alpha^n} + \overline{a_{n-1}}\,\overline{\alpha^{n-1}} + \cdots + \overline{a_0} = 0 \\ &\iff a_n \bar{\alpha}^n + a_{n-1}\bar{\alpha}^{n-1} + \cdots + a_0 = 0 \\ &\iff p(\bar{\alpha}) = 0 \end{aligned}$$

③ Factorization of polynomials over $\mathbb{C}$

Let $p(t) \in P(\mathbb{C})$, $p \neq 0$, $\deg p = n$. Then

$$p(t) = c(t - \alpha_1)(t - \alpha_2) \cdots (t - \alpha_n)$$

for some $c, \alpha_1, \cdots, \alpha_n \in \mathbb{C}$, $c \neq 0$.

④ Factorization of polynomials over $\mathbb{R}$

Let $p(t) \in P(\mathbb{R})$, $p \neq 0$, $\deg p = n$. Then

$$p(t) = c\, q_1(t) \cdots q_k(t)(t - \alpha_1) \cdots (t - \alpha_m)$$

for some irreducible $q_j(t) = t^2 + b_j t + c_j \in P_2(\mathbb{R})$

$c, \alpha_1, \cdots, \alpha_m \in \mathbb{R}$, $c \neq 0$ with $2k + m = n$

Rmk In ③ and ④, the factors are unique up to re-indexing

Pf ④ Let $p(t)$ has real roots $\alpha_1, \cdots, \alpha_m$,

non-real roots $\beta_1, \overline{\beta_1}, \beta_2, \overline{\beta_2}, \cdots, \beta_k, \overline{\beta_k}$

Let $q_j(t) = (t-\beta_j)(t-\overline{\beta_j})$

$$= t^2 - (\beta_j + \overline{\beta_j})t + \beta_j\overline{\beta_j}$$

$$= t^2 - 2\,\mathrm{Re}\,\beta_j\, t + |\beta_j|^2$$

Note $q_j(t) \in P_2(\mathbb{R})$ and is irreducible

So $p(t) = c(t-\alpha_1)\cdots(t-\alpha_m)\cdot$

$$(t-\beta_1)(t-\overline{\beta_1})\cdots(t-\beta_k)(t-\overline{\beta_k})$$

$$= c(t-\alpha_1)\cdots(t-\alpha_m)\, q_1(t)\cdots q_k(t)$$

eg $p(t) = t^4+1$ has roots

$\beta_1 = \frac{1}{\sqrt{2}}(1+i)$ $\quad \overline{\beta_1} = \frac{1}{\sqrt{2}}(1-i)$

$\beta_2 = \frac{1}{\sqrt{2}}(-1+i)$ $\quad \overline{\beta_2} = \frac{1}{\sqrt{2}}(-1-i)$

$p(t) = (t-\beta_1)(t-\overline{\beta_1})(t-\beta_2)(t-\overline{\beta_2})$ (over $\mathbb{C}$)

$\quad = (t^2 - \sqrt{2}t + 1)(t^2 + \sqrt{2}t + 1)$ (over $\mathbb{R}$)

Prop 7.26 Let $\dim V < \infty$, $T \in L(V)$ be self-adjoint, $q(t) = t^2 + bt + c \in P_2(\mathbb{R})$ be irreducible. Then $q(T)$ is invertible

Pf Let $v \in V$, $v \neq \vec{0}$. Then

$$\langle q(T)(v), v\rangle$$

$$= \langle (T^2 + bT + cI_V)(v), v\rangle$$

$T = T^*$

$\therefore \langle T(v), v\rangle \in \mathbb{R}$

$$= \langle T^2(v), v\rangle + b\langle T(v), v\rangle + c\langle v, v\rangle$$

$$\geq \langle T(v), T(v)\rangle - |b\langle T(v), v\rangle| + c\|v\|^2$$

$$\geq \|T(v)\|^2 - |b|\,\|T(v)\|\,\|v\| + c\|v\|^2$$

(Cauchy Schwarz)

$$= \left(\|T(v)\| - \frac{|b|}{2}\|v\|\right)^2 + \left(c - \frac{b^2}{4}\right)\|v\|^2$$

$$> 0$$

q irreducible $\Rightarrow b^2 - 4c < 0$

$\Rightarrow c - \frac{b^2}{4} > 0$

$\therefore\ q(T)(v) \neq \vec{0}$ for $v \neq \vec{0}$

$\therefore\ \operatorname{null} q(T) = \{0\}$

$q(T)$ is injective, $\dim V < \infty$

$\Rightarrow q(T)$ is invertible

Prop 7.27 Suppose $0 < \dim V < \infty$,
If $T \in \mathcal{L}(V)$ is self-adjoint,
then T has an eigenvalue.

Pf If $\mathbb{F} = \mathbb{C}$, T has an e.value by 5.21

Suppose $\mathbb{F} = \mathbb{R}$. Let $\dim V = n$, $\vec{0} \neq v \in V$

Then the $n+1$ vectors

$v, T(v), \cdots, T^n(v)$ are lin indept

$\Rightarrow \exists\ a_0, \cdots, a_n \in \mathbb{R}$, not all zero, s.t.

$$a_0 v + a_1 T(v) + \cdots + a_n T^n(v) = \vec{0}$$

Let $p(t) = a_0 + a_1 t + \cdots + a_n t^n \in P_n(\mathbb{R})$

$$= c\, q_1(t) \cdots q_k(t)(t-\lambda_1)\cdots(t-\lambda_m)$$

where $c, \lambda_1, \cdots, \lambda_m \in \mathbb{R}$, $c \neq 0$ and

$q_i(t) = t^2 + b_i t + c_i$ is irreducible, $i = 1, \cdots, k$

Note $p(T)(v) = 0$

$\Rightarrow p(T) = c\, q_1(T) \cdots q_k(T)(T - \lambda_1 I_V) \cdots (T - \lambda_m I_V)$

is not injective

$c \neq 0$, $q_i(T)$ is invertible for each i by 7.26

$\Rightarrow T - \lambda_j I_V$ is not injective for some j

$\Rightarrow \lambda_j$ is an e.value of T

Thm 7.29 (Real Spectral Theorem)

Let $\mathbb{F} = \mathbb{R}$, $\dim V < \infty$, $T \in L(V)$.

Then T is self-adjoint

$\Leftrightarrow$ T has an orthonormal eigenbasis

Pf ($\Leftarrow$)

Let $\beta = \{e_1, \cdots, e_n\}$ be orthonormal e.basis.

with $T(e_i) = \lambda_i e_i$,

Let $A = M(T, \beta) = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

Then

$$M(T^*, \beta) = A^* = \begin{bmatrix} \overline{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \overline{\lambda_n} \end{bmatrix} = A$$

$\Rightarrow T^* = T$

($\Rightarrow$) We prove it by induction on $n = \dim V$.

Suppose T is self-adjoint

For $n = 0$, T has an orthonormal eigenbasis ϕ

Assume we have proved ($\Rightarrow$) for $n \leq k$

Suppose $n = k+1$. By 7.21, T has an eigenvalue λ_1

Let e_1 be a corr. e.vector with $\|e_1\| = 1$

Let $U = \operatorname{span}\{e_1\}$. For any $k \in \mathbb{R}$,

$T(ke_1) = kT(e_1) = k\lambda_1 e_1 \in U \Rightarrow U$ is T-invariant

By 7.28, $U^\perp$ is T-invariant and

$T|_{U^\perp}$ are self-adjoint, $\dim U^\perp = n-1 = k$

Induction assumption $\Rightarrow T|_{U^\perp}$ has orthonormal e.basis $\{e_2, \cdots, e_n\}$

$V = U \oplus U^\perp$, $U \perp U^\perp$, each e_i is e.vector, $\|e_i\| = 1$

$\Rightarrow \{e_1, \cdots, e_n\}$ is an orthonormal eigenbasis of T $\blacksquare$